2. THEORY OF EQUATIONS

Quick Review

1. An expression of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where n is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers such that $a_n \neq 0$ is called a ***polynomial*** in x of degree n. A complex number α is said to be a ***zero*** of the polynomial f(x) if $f(\alpha) = 0$.
2. An n^{th} degree polynomial is also represented as $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ where $p_0, p_1, p_2, \ldots p_n$ are complex numbers and $p_0 \neq 0$.
3. The polynomial $f(x) = x^n + p_1 x^{n-1} + p_1 x^{n-2} + \ldots + p_n$ is called the ***standard form*** of n th degree polynomial.
4. Polynomials of degree 0, 1, 2, 3, 4 are respectively called as a ***constant, linear, quadratic, cubic, biquadratic*** polynomials.
5. Degree of a polynomial f(x) is denoted by deg f(x).
6. If f(x) and g(x) are two polynomials such that deg f(x) = m, deg g(x) = n then deg [f(x) g(x)] = m+n.
7. **Division Algorithm :** If f(x), $0 \neq g(x)$ are two polynomials, then $\exists$ polynomicals q(x), r(x) uniquely such that f(x) = q(x) g(x) + r(x) where r(x) = 0 or deg r(x) < deg g(x).
8. The polynomial q(x) is called quotient and the polynomial r(x) is called remainder of f(x) when divided by g(x).
9. **Remainder Theorem :** If f(x) is a polynomial, then the remainder of f(x) when divided by x – a is f(a).
10. Let f(x), g(x) be two polynomials. Then g(x) is said to ***divide*** f(x) or g(x) is said to be a ***divisor*** or ***factor*** of f(x) if there exists a polynomial q(x) such that f(x) = q(x) g(x).
11. **Factor theorem :** If f(x) is a polynomial then f(a)=0 iff (x – a) is a factor of f(x).
12. **Taylor's Theorem :** If f(x) is a polynomial function of degree n, then

$$f(x + h) = f(h) + \frac{x}{1!}f'(h) + \frac{x^2}{2!}f''(h) + \ldots + \frac{x^n}{n!}f^{(n)}(h).$$

13. **Horner's Method :** If f(x) is a polynomial function of degree n then

$f(x + h) = q_0x^n + q_1x^{n-1} + q_2x^{n-2} + \ldots + q_n$ where

$q_1, q_2, q_3, \ldots, q_n$ are the remainders of f(x) when divided by $(x - h)^n, (x - h)^{n-1}, \ldots, x - h$ respectively and q_0 = coefficient of x^n in f(x).

14. An equation f(x) = 0 is said to be an ***algebraic equation*** or a ***polynomial equation*** of an ***equation*** of degree n if f(x) is a polynomial of degree n. A complex number α is said to be a ***root*** of the equation f(x) = 0 if $f(\alpha) = 0$.
15. **Fundamental Theorem of Algebra :** Every algebraic equation of nonzero degree has a root.
16. The equation having roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ is $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n) = 0$.
17. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$, then

    1) Sum of the roots = $\sum \alpha_1 = s_1 = -\frac{p_1}{p_0}$

    2) Sum of the products of roots taken two at a time = $\sum \alpha_1\alpha_2 = s_2 = \frac{p_2}{p_0}$

3) Sum of the products of roots taken three at a time = $\sum \alpha_1\alpha_2\alpha_3 = s_3 = -\frac{p_3}{p_0}$ …….

n) Product of the roots = $\alpha_1\,\alpha_2\ldots\alpha_n = s_n = (-1)^n \frac{p^n}{p_0}$.

18. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then
i) $\alpha + \beta + \gamma = s_1 = -b/a$ ii) $\alpha\beta + \beta\gamma + \gamma\alpha = s_2 = c/a$ iii) $\alpha\beta\gamma = s_3 = -d/a$

19. If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then
i) $\alpha + \beta + \gamma + \delta = s_1 = -b/a$ ii) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = s_2 = c/a$
iii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = s_3 = -d/a$ iv) $\alpha\beta\gamma\delta = s_4 = e/a$

20. The equation having roots α, β is $x^2-(\alpha+\beta)x+\alpha\beta=0 \Rightarrow x^2 - s_1x + s_2 = 0$.

21. The equation having roots, α, β, γ is $x^3 - (\alpha + \beta + \gamma)\,x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - (\alpha\beta\gamma) = 0 \Rightarrow x^3 - s_1x^2 + s_2x - s_3 = 0$.

22. The equation having roots, $\alpha, \beta, \gamma, \delta$ is $x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + (\alpha\beta\gamma\delta) = 0 \Rightarrow x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$.

23. A root α of an algebraic equation $f(x) = 0$ is said to be a ***multiple root*** of order m if it occurs m times.

24. A multiple root α of order m of $f(x) = 0$ is a multiple root of order $m - 1$ of $f'(x) = 0$ where $f'(x)$ is the derivative of $f(x)$.

25. **Newton's Theorem :** If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ and $S_r = \alpha_1^r + \alpha_2^r + \ldots + \alpha_n^r$ then
i) $S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + p_{r-1}S_1 + r.p_r = 0$ when $r \le n$.
ii) $S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + p_nS_{r-n} = 0$ when $r > n$.

26. **Horner's Theorem :** Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ and $S_r = \alpha_1^r + \alpha_2^r + \ldots + \alpha_n^r$
i) If $r \ge 0$ then S_r is the coefficient of $\frac{1}{x^{r+1}}$ in $\frac{f'(x)}{f(x)}$ when it is expressed in powers of $1/x$.
ii) If $r < 0$, $\alpha_i \ne 0$ for $i = 1, 2, 3, \ldots n$ then S_r is the coefficient of $x^{-(r+1)}$ in $\frac{-f'(x)}{f(x)}$ when it is expressed in powers of x.

27. If $r \ge 0$, then S_{-r} = Coefficient of x^{r-1} in $\frac{f'(x)}{f(x)}$ when it is expressed in powers of x.

28. If the roots of $f(x) = 0$ are in H.P. then the roots of $f(1/x) = 0$ are in A.P.

29. In an equation with real coefficients, imaginary roots occur in conjugate pairs.

30. In an equation with rational coefficients, irrational roots occur in pairs of conjugate surds.

31. In an equation, if all the coefficients are positive, then the equation has no positive root.

32. In an equation, if all the coefficients of the even powers of x are of one type (either positive or negative) and the coefficients of odd powers of x are of opposite sign, then the equation has no negative root.

33. In an equation if all the powers of x are odd and all the coefficients are of the same sign, then the equation has no real root except 0.

34. If $f(x)$ is a polynomial with real coefficients and $\alpha > 0$ then the number of changes of sign of $(x - \alpha)\,f(x)$ is more than the number of changes of sign in $f(x)$.

35. **Descarte's Rule of Signs :** An equation f(x) = 0 can not have more positive roots than there are changes of sing in f(x) and cannot have more negative roots than there are changes of sign in f(–x).
36. If f(x) is a polynomial such that f(a) and f(b) have opposite signs then one root of f(x) = 0 must lie between a and b.
37. The equation whose roots are those of the equation f(x) = 0 with contrary signs is f(–x) = 0.
38. The equation whose roots are the multiples of k ($\neq 0$) of those of the proposed equation f(x) = 0 is f(x/k) = 0.
39. The equation whose roots are the reciprocals of the roots of f(x) = 0 is f(1/x) = 0.
40. The equation whose roots are exceed by h than those of f(x) = 0 is f(x – h) = 0.
41. If $f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ then to eliminate the second term f(x) = 0 can be transformed to f(y + h) = 0 where $h = -\frac{p_1}{np_0}$.
42. The equation whose roots are the squares of the roots of f(x) = 0 is obtained by eliminating square root from $f(\sqrt{x}) = 0$.
43. The equation whose roots are the cubes of the roots of f(x) = 0 is obtained by eliminating cube root from $f(\sqrt[3]{x}) = 0$.
44. An equation f(x) = 0 is said to be a ***reciprocal equation*** if $1/\alpha$ is a root of f(x) = 0 whenever α is a root of f(x) = 0.
45. An equation $f(x) = p_0 x^n + p_1 x^{n-1} + \ldots + p_n = 0$ is a reciprocal equation iff either $p_i = p_{n-i}$ for every i, or $p_i = -p_{n-i}$ for every i.
46. A reciprocal equation $f(x) = p_0 x^n + p_1 x^{n-1} + \ldots + p_n = 0$ is said to be a ***reciprocal equation of first class*** if $p_i = p_{n-i}$, for all i.
47. A reciprocal equation $f(x) = p_0 x^n + p_1 x^{n-1} + \ldots + p_n = 0$ is said to be a ***reciprocal equation of second class*** if $p_i = -p_{n-i}$, for all i.
48. If f(x) = 0 is a reciprocal equation of degree n, then $x^n f(1/x) = \pm f(x)$.
49. If f(x) = 0 is a reciprocal equation of first class and odd degree then –1 is a root of f(x) = 0.
50. If f(x) = 0 is a reciprocal equation of second class and of odd degree then 1 is a root of f(x) = 0.
51. If f(x) = 0 is a reciprocal equation of second class and of even degree then 1, –1 are roots of f(x) = 0.
52. The equation of lowest degree with rational coefficients, having a root

    i) $\sqrt{a} + \sqrt{b}$ is $x^4 - 2(a+b)x^2 + (a-b)^2 = 0$ ii) $\sqrt{a} + \sqrt{bi}$ is $x^4 - 2(a-b)x^2 + (a+b)^2 = 0$
53. The condition that the roots of $ax^3 + bx^2 + cx + d = 0$ may be in

    i) A.P. is $2b^3 + 27a^2d = 9abc$

    ii) G.P. is $ac^3 = b^3d$

    iii) H.P. is $2c^3 + 27ad^2 = 9bcd$
54. The condition that one root of $ax^3 + bx^2 + cx + d = 0$ may be the sum of the other two roots is $8a^2d + b^3 = 4abc$.
55. The condition that the product of two of the roots of $ax^3+bx^2+cx+d = 0$ may be –1 is $a(a+c)+d(b+d) = 0$.
56. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$ then

i) $\alpha^2 + \beta^2 + \gamma^2 = s_1^2 - 2s_2 = \dfrac{b^2 - 2ac}{a^2}$

ii) $\alpha^3+\beta^3+\gamma^3 = s_1^3 - 3s_1s_2 + 3s_3 = \dfrac{3abc - b^3 - 3a^2d}{a^3}$

iii) $\alpha^4 + \beta^4 + \gamma^4 = s_1^4 - 4s_1^2s_2 + 4s_1s_3 + 2s_2^2 = \dfrac{b^4 - 4ab^2c + 4a^2bd + 2a^2c^2}{a^4}$

iv) $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta) = 2c/a$.

57. If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4+bx^3+cx^2+dx+e = 0$ then

i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = s_1^2 - 2s_2 = \dfrac{b^2 - 2ac}{a^2}$

ii) $\alpha^3+\beta^3+\gamma^3+\delta^3 = s_1^3 - 3s_1s_2 + 3s_2 = \dfrac{3abc - b^3 - 3a^2d}{a^3}$

iii) $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = s_1^4 - 4s_1^2s_2 + 4s_1s_3 + 2s_2^2 - 4s_4 = \dfrac{b^4 - 4ab^2c + 4a^2bd + 2a^2c^2 - 4a^3e}{a^4}$

iv) $\Sigma\, \alpha^2\beta = s_1s_2 - 3s_3$.

v) $\Sigma\, \alpha^2\beta\gamma = s_1s_3 - 4s_4$.

58. If α, β, γ are the roots of $f(x) = x^3 + px^2 + qx + r = 0$ then the equation having roots

i) $\beta + \gamma, \gamma + \alpha, \alpha + \beta$ is $f(-p - x) = 0$.

ii) $\beta\gamma, \gamma\alpha, \alpha\beta$ is $f\left(-\dfrac{r}{x}\right) = 0$.

iii) $\beta^2\gamma^2, \gamma^2\alpha^2, \alpha^2\beta^2$ is $f\left(\dfrac{r}{\sqrt{x}}\right) = 0$.

iv) $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$ is $f\left(\dfrac{r}{x - q}\right) = 0$.

v) $\beta\gamma + \dfrac{1}{\alpha}, \gamma\alpha + \dfrac{1}{\beta}, \alpha\beta + \dfrac{1}{\gamma}$ is $f\left(\dfrac{1 - r}{x}\right) = 0$.

vi) $\alpha - \dfrac{1}{\beta\gamma}, \beta - \dfrac{1}{\gamma\alpha}, \gamma - \dfrac{1}{\alpha\beta}$ is $f\left(\dfrac{rx}{r + 1}\right) = 0$.